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THE SECOND-ORDER
RYTOV APPROXIMATION

H. T. Yura

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PREFACE

This Memorandum, prepared as part of RAND's continuing study of long-distance optical propagation through the atmosphere, gives a workable solution to the second-order Rytov approximation.

Recently there has been a controversy on the ranges for which the first-order Rytov approximation gives an adequate engineering estimate of the effect of a turbulent medium on laser propagation. Therefore, it is felt that an explicit derivation of the second-order Rytov approximation should be of interest to those concerned with optical communication and laser radar. Furthermore, the second-order approximation is necessary because it is the lowest order nontrivial approximation which conserves average energy to the order of the approximation.

SUMMARY

An explicit and useful formulation of the solution for the second-order Rytov approximation is given. From this solution a condition of validity for the Rytov solution is obtained. It is concluded that, in general, both the Born and Rytov approximations have the same domain of validity.

CONTENTS

I. INTRODUCTION

A great deal of theoretical effort has been given to calculations based on the so-called Rytov approximation.[1-3] However, the validity of the Rytov approximation has been questioned by a number of authors.[2,4] In a recent paper Fried[5] continues to take issue with the arguments of Hufnagel and Stanley[4] and Brown,[2] who claim that the Born and Rytov approximations have the same domain of validity. In view of this continuing controversy, it is felt that an explicit derivation should be given of the second-order Rytov approximation (defined below). In this Memorandum the hierarchy of Rytov equations is presented. The explicit solution (in quadrature) of the second-order Rytov approximation is obtained. In Ref. 6 the Born approximation was examined in detail, and it was assumed that the domain of validity of the Rytov solution was the same. Here we present an explicit demonstration that this is indeed the case.

The solution obtained is correct through terms of second order in n_1, where n_1 is the fluctuating part of the index of refraction. From this solution a condition of validity is obtained for the Rytov approximation with the conclusion that both the Born and Rytov approximations have the same domain of validity.

As in Ref. 6, the analysis is time independent and restricted to weakly inhomogeneous media which are assumed to be statistically homogeneous and isotropic, characterized by an index of refraction correlation function. Furthermore, the electrical conductivity and magnetic permeability of the medium are taken as zero and one, respectively. Scalar waves at optical wavelengths are considered, with the extensions

to vector fields being straightforward. For further details of the theory of optical propagation through a turbulent medium, the reader is referred to Ref. 6 and the references therein.

The scalar wave equation is

$$\nabla^2 U + k^2 n^2(\vec{r})U = 0 \tag{1}$$

where U is a typical component of the field, k is the optical wave number, and

$$n(\vec{r}) = 1 + n_1(\vec{r}) \tag{2}$$

where n_1 is the fluctuating part of the index of refraction assumed to satisfy $|n_1| \ll 1$ and $\overline{n_1} = 0$ (the bar over a quantity indicates the ensemble average of the quantity).

II. THE SECOND-ORDER RYTOV APPROXIMATION

The Rytov transformation consists of setting $U = \exp[\psi]$ in Eq. (1) and thus obtaining

$$\nabla^2 \psi + (\nabla\psi)^2 + k^2 [1 + n_1(\vec{r})]^2 = 0 \qquad (3)$$

In the literature it is customary[1-3] to seek a solution for ψ as a power series in n_1. Let

$$\psi = \sum_{m=0}^{\infty} \psi_m \qquad (4)$$

where ψ_0 is zero order in n_1, ψ_1 is first order in n_1, and so forth. Then it can be shown that the ψ_m's satisfy

$$\nabla^2 \psi_0 + (\nabla\psi_0)^2 + k^2 = 0 \qquad (5)$$

$$\nabla^2 \psi_1 + 2\nabla\psi_0 \cdot \nabla\psi_1 + 2k^2 n_1(\vec{r}) = 0 \qquad (6)$$

$$\nabla^2 \psi_2 + 2\nabla\psi_0 \cdot \nabla\psi_2 + k^2 n_1^2(\vec{r}) + (\nabla\psi_1)^2 = 0 \qquad (7)$$

and

$$\nabla^2 \psi_m + 2\nabla\psi_0 \cdot \nabla\psi_m + \sum_{p=0}^{m-1} \nabla\psi_p \cdot \nabla\psi_{m-p} = 0, \quad m = 3,4,5\ldots \qquad (8)$$

Equation (5) gives the solution to Eq. (1) for the case $n_1 = 0$. Thus if $U_0 = \exp \psi_0$, then Eq. (5) is equivalent to $\nabla^2 U_0 + k^2 U_0 = 0$.

The function ψ_1 is what is usually referred to in the literature[1] as the Rytov approximation. In this Memorandum, ψ_1 is referred to as

the first-order Rytov approximation, while ψ_2 is referred to as the
second-order Rytov approximation.

It can be shown[1] that within an arbitrary constant the general
solution of Eq. (6) is given by

$$\psi_1 = - \frac{2k^2}{U_0(\vec{r})} \int d\vec{r}\,' \; G(\vec{r} - \vec{r}\,') \; n_1(\vec{r}\,') \; U_0(\vec{r}\,') \tag{9}$$

where the integration in Eq. (9) extends over the region of space
where $n_1(\vec{r})$ is different from zero and

$$G(\vec{r} - \vec{r}\,') = - \frac{1}{4\pi} \left[\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \right] \tag{10}$$

is the Green's function in the absence of turbulence, which satisfies
the outgoing radiation condition at infinity.

Others have failed to obtain a useful workable solution to
Eq. (7) (e.g., Schmeltzer[3]). It is noted that Eq. (7) can be solved
by the use of a transformation suggested by a comparison of the Born
and Rytov expansions. Upon equating second-order terms in the expansions
we have

$$\psi_2 = \varphi_2 - \frac{\psi_1^2}{2} \tag{11}$$

where ψ_1 is given in Eq. (9). Upon substituting Eq. (11) into Eq.
(7) and using Eq. (6), we find

$$\nabla^2 \varphi_2 + 2\nabla U_0 \cdot \nabla \varphi_2 = - (k^2 n_1^2 + 2k^2 n_1 \psi_1) \tag{12}$$

Let

$$\phi_2(\vec{r}) = \frac{W(\vec{r})}{U_0(\vec{r})} \tag{13}$$

and substitute this into Eq. (12). We find that W satisfies

$$\nabla^2 W + k^2 W = - (k^2 n_1^2 + 2k^2 n_1 \phi_1) U_0$$

Hence,

$$W(\vec{r}) = - k^2 \int G(\vec{r} - \vec{r}') \left[n_1^2(\vec{r}') + 2n_1(\vec{r}') \phi_1(\vec{r}') \right] U_0(\vec{r}') \, d\vec{r}' \tag{14}$$

Thus, the second-order Rytov approximation is given by Eq. (11), where ϕ_2 and W are given by Eqs. (13) and (14), respectively.

The transformation employed to solve Eq. (7) suggests that the solution to Eq. (14) should be equal to the second-order Born approximation. The Born expansion for $U(\vec{r})$ is given by

$$U(\vec{r}) = \sum_{i=0}^{\infty} U_i(\vec{r}) \tag{15}$$

where

$$\left(\nabla^2 + k^2 \right) U_0 = 0 \tag{16}$$

$$U_1(\vec{r}) = - 2k^2 \int d\vec{r}' \, G(\vec{r} - \vec{r}') \, n_1(\vec{r}') \, U_0(\vec{r}') \tag{17}$$

and for $i > 1$

$$U_{i+1}(\vec{r}) = -2k^2 \int d\vec{r}' \; G(\vec{r} - \vec{r}') \; n_1(\vec{r}') \; U_i(\vec{r}')$$

$$-k^2 \int d\vec{r}' \; G(\vec{r} - \vec{r}') \; n_1^2(\vec{r}') \; U_{i-1}(\vec{r}') \tag{18}$$

For $i = 1$ Eq. (18) gives

$$U_2(\vec{r}) = -2k^2 \int d\vec{r}' \; G(\vec{r} - \vec{r}') \; n_1(\vec{r}') \; U_1(\vec{r}')$$

$$-k^2 \int d\vec{r}' \; G(\vec{r} - \vec{r}') \; n_1^2(\vec{r}') \; U_0(\vec{r}') \tag{19}$$

Comparing Eq. (14) with Eq. (19), it is seen that to within an arbitrary constant $W(\vec{r}) = U_2(\vec{r})$. Hence, to within an arbitrary constant the second-order Rytov approximation is given by

$$\psi_2(\vec{r}) = \frac{U_2(\vec{r})}{U_0(\vec{r})} - \frac{\psi_1^2(\vec{r})}{2} = \frac{U_2(\vec{r})}{U_0(\vec{r})} - \frac{1}{2}\left[\frac{U_1(\vec{r})}{U_0(\vec{r})}\right]^2 \tag{20}$$

The average field is determined, to terms of second order in n_1, as

$$\overline{U} = \overline{\exp\left[\psi_0 + \psi_1 + \psi_2\right]}$$

$$= \exp\left[\psi_0 + \overline{\psi}_2\right] \tag{21}$$

where $\overline{\psi}_2(\vec{r}) = \overline{\left[U_2(\vec{r})/U_0(\vec{r})\right]}$. For example, when the field in the absence of turbulence is a plane wave

$$\overline{\psi}_2 = i\vec{k} \cdot \vec{r} \left\{ \frac{B_n(0)}{2} + \left(\frac{k^2}{4\pi}\right) \int d\vec{R} \; \frac{e^{ikR}}{R} \; B_n(\vec{R}) \; e^{i\vec{k}\cdot\vec{R}} \right\} \tag{22}$$

where $B_n(\vec{R}) = \overline{n_1(\vec{r}) \, n_1(\vec{r} + \vec{R})}$ is the correlation function of the index of refraction fluctuations.

It is shown in Ref. 6 that one must include the second-order contribution to the field (ψ_2) in order to obtain an adequate engineering approximation to the scattered field in the presence of turbulence. Furthermore, a second-order expansion for the field is the lowest order nontrivial approximation which conserves average energy to the order of the approximation. That is, the field

$$U(\vec{r}) = \exp\left[\psi_0(\vec{r}) + \psi_1(\vec{r}) + \psi_2(\vec{r})\right] \tag{23}$$

which contains all terms through second order in n_1, gives the correct average field, correct phase and intensity statistics, conserves average energy, and satisfies the optical theorem to this order. In the present notation the optical theorem is, through terms of second order in n_1, given by

$$2\mathrm{Re}\psi_2 + |\psi_1|^2 = 0 \tag{24}*$$

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* In the usual notation the optical theorem is: $\sigma = \dfrac{4\pi}{k}\,\mathrm{Im}f(0)$, where σ is the total scattering cross section per unit volume and $f(0)$ is the forward scattering amplitude per unit volume.

III. VALIDITY CONDITION

To obtain the condition of validity for the Rytov solution $U = \exp\left[\psi_0 + \psi_1 + \psi_2 + \cdots\right]$, it is sufficient to require that each successive term in the expansion of ψ be smaller than the preceding term.[2] In view of this, and since the various terms in the expansion of ψ are random functions which can be described only in statistical terms, it is assumed that the condition of validity on the Rytov solution is related to the relative magnitude of the square value of successive terms. Therefore, the condition of validity for the Rytov solutions is taken to be

$$(\psi_2)^2 \ll (\psi_1)^2 \qquad (25)$$

where $\psi_2 = \varphi_2 - \dfrac{\psi_1^2}{2}$. Now, it can be shown that $\left|\mathrm{Im}\,\varphi_2\right| \ll \left|\mathrm{Re}\,\varphi_2\right|$ for all cases of interest: hence, by expressing $\mathrm{Re}\,\varphi_2$ in terms of ψ_1 by using Eq. (24), we find from Eq. (25) that two conditions must be satisfied (from the real and imaginary parts, respectively).

The real part yields the condition $(\mathrm{Re}\psi_1)^2 \ll 1$, while the imaginary part yields the stronger condition, since it can be shown[*] that $(\mathrm{Re}\psi_1)^2 \leq (\mathrm{Im}\,\psi_1)^2$

$$\frac{1}{2}\left|\psi_1\right|^2 \ll 1 \qquad (26)$$

Taking the ensemble average of Eq. (26), we conclude that the condition of validity of the Rytov solution is given by

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[*]See Ref. 1, Chap. 7 and 9.

$$\overline{\frac{1}{2} \, |\psi_1|^2} \ll 1 \tag{27}$$

It is noted that when $\psi_2^2 \sim \psi_1^2$, the successive terms in the asymptotic expansion for ψ (i.e., $\psi = \psi_0 + \psi_1 + \psi_2 + \cdots$) will all be of the same order of magnitude. The solution given by Eq. (23) is therefore valid only when Eq. (27) is satisfied.

The condition of validity of the Born approximation is given by exactly the same expression (Eq. (57) of Ref. 6).

We conclude that, in general, both the Born and Rytov approximations have the same domain of validity. A critical range R_c is determined by solving Eq. (27) with an equality sign. For $R < R_c$, the Born and Rytov solutions are valid, while for $R \gtrsim R_c$ both solutions fail to yield a valid solution. Treatments which apply the Rytov approximation beyond R_c, and therefore predict large dispersion of intensity,[7] are incorrect.

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10. ABSTRACT

An explicit and useful formulation of the solution of the second-order Rytov approximation for estimating the effect of a turbulent medium on laser propagation. The first-order Rytov approximation has been much used in theoretical analyses but recently there has been a controversy on the ranges for which it gives an adequate estimate. RM-5697-PR showed that the calculations of the EM field must include all the second-order terms of the fluctuating part of the index of refraction, proved that the Born approximation is valid only over a limited range (a few kilometers at optical wavelengths), and assumed that the same validity condition applied to the Rytov approximation; this study explicitly demonstrates that this is the case. The second-order Rytov approximation is obtained by equating the second-order terms in the Rytov and Born expansions and is explicitly solved in quadrature. The field equation that results gives the correct average field, phase and intensity statistics, conserves average energy, and satisfies the optical theorem--within the domain of validity, which is the same for the Rytov and the Born approximations. Analyses that apply the Rytov approximation over longer distances, and therefore predict large dispersion, are incorrect.

11. KEY WORDS

Turbulence
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